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# AN ELEMENTARY EXPOSITION OF THE THEORY OF FINITE DIFFERENCES.*

By SAUL EPSTEEN.

## §1.

In the Differential Calculus we study the *limit* of the ratio$[f(x+\Delta x)-f(x)]\div\Delta x$ as $\Delta x$ approaches 0; in the Calculus of Finite Differences however we study it as a ratio in which $\Delta x$ is finite.

The function of $x$ under consideration being $u$, it is customary to write it $u_x$ and to designate its increment by $\Delta u_x$ where $\Delta u_x=u_{x+\Delta x}-u_x$. In the Differential Calculus $\frac{du}{dx}$ is not a fraction but a symbol of operation, $du$, $dx$ having no meanings by themselves. In the Calculus of Finite Differences on the other hand $\frac{\Delta u_x}{\Delta x}$ is a true fraction.

Instead of considering $\frac{\Delta u_x}{\Delta x}$, the following simplification is ordinarily introduced. Letting $\Delta x=h$ and writing $x=ht$ it follows at once that for $h=$constant, $\Delta x=h\Delta t$, whence $\Delta t=1$. Thus it is seen that by a simple transformation of the real variable $x$, the increment may be taken as unity; replacing $t$ by $x$, $\Delta x=1$ and the attention need now be directed merely to the operation $\Delta$, defined by the equation $\Delta u_x=u_{x+1}-u_x$. The above simplification consists at bottom of selecting the finite increment $\Delta x$ as the unit of measure.

## §2.

Since $\Delta u_x$ is itself a function of $x$ its increment $\Delta(\Delta u_x)$ may be found by the same process as above; it is ordinarily designated by $\Delta^2 u_x$. In a like manner the differences $\Delta^2 u_x$, $\Delta^3 u_x$, ......... may be found. As an example of this notion of successive differences consider the very simple function $u_x=x^2$. Evidently $\Delta u_x=u_{x+1}-u_x=2x+1$; $\Delta^2 u_x=2$. The following table may therefore be made:

| value of $x$ | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|
| $u_x$ | 1 | 4 | 9 | 16 | 25..... |
| $\Delta u_x$ | 3 | 5 | 7 | 19........ | .......... |
| $\Delta^2 u_x$ | 2 | 2 | 2........ | .......... | .......... |

It is interesting to observe that if any term be subtracted from the one immediately following to the right, the difference will be the term immediately below it. This is general of course, being due to the fact that $\Delta u_x=u_{x+1}-u_x$.

It will now be shown that if $u_x=ax^n$, $\Delta^n u_x=an!$

By definition, $\Delta u_x=a(x+1)^n-ax^n=ax^{n-1}+b_1x^{n-2}+b_2x^{n-3}+\dots\dots$, $b_1$, $b_2$,

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*Literature: A Treatise on the Calculus of Finite Differences, by George Boole; Theorie der Differenzen und Summen, by O. Schloemilch; Differenzenrechnung, by A. A. Markoff; Differenzenrechnung, by D. Seliwanoff in the *Encyklopaedia der Mathematischen Wissenschaften*, Bd. 1, Heft 6.

$\dots\dots$, being constants; repeating the process the result may be written $\Delta^2 u_x = an(n-1)x^{n-2} + c_1 x^{n-3} + c_2 x^{n-4} + \dots\dots$, $c_1, c_2, \dots\dots$ being constants; continuing in this manner the desired result is reached.

Evidently $\Delta^r a x^s = 0$ if $r > s$, hence

$$\Delta^n (a_1 x^n + a_2 x^{n-1} + \dots\dots + a_{n-1} x + a_n) = a_1 n!$$

An expression

$$x(x-1)(x-2) \dots\dots (x-m+1) \equiv x^{(m)}$$

is known as a *factorial.* It is important to observe that *in the Calculus of Finite Differences factorials play an analogous rôle to that of powers of x in the Differential Calculus.*

Thus, $\dfrac{dx^m}{dx} = mx^{m-1}$; now $\Delta x^{(m)} = (x+1)^{(m)} - x^{(m)}$

$$= (x+1)x(x-1) \dots\dots (x-m+2) - x(x-1) \dots\dots (x-m+1) = mx^{(m-1)}.$$

The following theorem of the Calculus of Finite Differences reminds one of Maclaurin's theorem in Differential Calculus: Let $\phi(x)$ be a rational integral function of $x$ of the $m$th degree, then it may be expanded in a series of factorials thus:

$$(1) \quad \phi(x) = \phi(0) + \Delta(0).x + \frac{\Delta^2 \phi(0)}{2!} x^{(2)} + \frac{\Delta^3 \phi(0)}{3!} x^{(3)} + \dots + \frac{\Delta^m \phi(0)}{m!} x^{(m)}.$$

In proof, assume

$$\phi(x) = a + bx + cx^{(2)} + dx^{(3)} + \dots\dots + hx^{(m)},$$

then

$$\Delta \phi(x) = b + 2cx + 3dx^{(2)} + \dots\dots + mhx^{(m-1)},$$

$$\Delta^2 \phi(x) = 2c + 3.2dx + \dots\dots + m(m-1)hx^{(m-2)},$$

$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$

$$\Delta^m \phi(x) = m(m-1) \dots\dots 2.1.h.$$

From this it is clear that

$$a = \phi(0),\ b = \Delta \phi(0),\ c = \frac{\Delta^2 \phi(0)}{2!},\ \dots\dots,\ h = \frac{\Delta^m \phi(0)}{m!}.$$

## §3.

In §6 difference equations will be studied of the general form

$$(2) \qquad \varphi_1(x, u_x, u_{x+1}, \dots\dots, u_{x+n}) = 0.$$

The considerations of §§1, 2 would lead one to suppose however that a linear difference equation should have the form

$$(3) \qquad \varphi_2(x,\ \Delta u_x,\ \Delta^2 u_x,\ \ldots\ldots,\ \Delta^n u_x)=0$$

where $\Delta^2 u_x = \Delta(\Delta u_x),\ \Delta^3 u_x = \Delta(\Delta^2 u_x),\ \ldots\ldots$

The equations (2) and (3) are in reality equivalent. First, by demonstrating that for any integer $j$ the difference $\Delta^j u_x$ can be expressed linearly with constant coefficients in terms of $u_{x+j},\ u_{x+j-1},\ \ldots\ldots,\ u_{x+1},\ u_x$ it will be concluded that an equation of the form (3) may be replaced by one of the form (2); secondly, by showing that $u_{x+j}$ can be expressed linearly with constant coefficients in terms of $\Delta^j u_x,\ \Delta^{j-1}u_x,\ \ldots\ldots,\ \Delta u_x,\ u_x$ it will be concluded that an equation of the form (2) can be replaced by one of the form (3).

I. By definition $\Delta u_x = u_{x+1} - u_x$, whence

$$\Delta^2 u_x = \Delta(\Delta u_x) = (u_{x+2} - u_{x+1}) - (u_{x+1} - u_x) = u_{x+2} - 2u_{x+1} + u_x.$$

By the ordinary method of induction it may be seen that

$$(4) \qquad \Delta^j u_x = u_{x+j} - j u_{x+j-1} + \frac{j(j-1)}{1.2} u_{x+j-2} + \ldots\ldots + (-1)^j u_x.$$

II. From $\Delta u_x = u_{x+1} - u_x$ it is seen that $u_{x+1} = u_x + \Delta u_x$, whence

$$u_{x+2} = u_x + \Delta u_x + \Delta(u_x + \Delta u_x) = u_x + 2\Delta u_x + \Delta^2 u_x.$$

It follows now by induction that

$$(5) \qquad u_{x+j} = u_x + j\Delta u_x + \frac{j(j-1)}{1.2}\Delta^2 u_x + \ldots\ldots + \Delta^j u_x.$$

## §4. APPLICATIONS.

I. Writing 0 for $x$ and $x$ for $j$ in equation (5) there results

$$(5') \qquad u_x = u_0 + x\Delta u_0 + \frac{x(x-1)}{2!}\Delta^2 u_0 + \ldots\ldots.$$

Suppose now there is given log3.14=.4969296, log3.15=.4983106, log3.16 =.4996871, log3.17=.5010593; required an approximate value of log3.14159.

The following table with the decimal point omitted may be written down at once:

| | $u_0$ | $u_1$ | $u_2$ | $u_3$ |
|---|---|---|---|---|
| | 4969296 | 4983106 | 4996871 | 5010593 |
| $\Delta$ | 13810 | 13765 | 13722 | |
| $\Delta^2$ | $-45$ | $-43$ | | |
| $\Delta^3$ | 2. | | | |

The common difference of 3.14, 3.15, 3.16, 3.17 is .01; this being taken as the unit gives the value of $x$ which corresponds to 3.14159 as .159. Hence by (5'),

$$(6)\quad u_x=4969296+.159\times 13810+\tfrac{1}{2}(.159)(.159-1)\times(-45)$$
$$+\tfrac{1}{6}(.159)(.159-1)(.159-2)\times 2;$$

whence $u_x=.4971495$, which is correct to the seventh decimal place.

II. In Life Insurance it is customary to denote the number of men living at an age $x$ by $l_x$. Suppose there is given a Mortality Table, obtained from statistics, containing the values of $l$ for intervals of five years; the problem is to interpolate for the intermediary values. The formula (5′) is applied.

Let $u_x$ be the desired value of the number living at the age $x$, $l_{x\pm z}=l'_z$ the number living at the nearest five-year period which is $z$ years younger or older than the age at which the value is desired. Evidently $z \ngtr 2$.

If $u_0=l'_{z-5}$, $u_1=l'_z$, $u_2=l'_{z+5}$, $u_x=l_x$, it is seen that

$$\triangle u_0=l'_z-l'_{z-5},\quad \triangle u_1=l'_{z+5}-l'_z, \text{ and}$$
$$\triangle^2 u_0=(l'_{z+5}-l'_z)-(l'_z-l'_{z-5})=l'_{z+5}-2l'_z+l'_{z-5}.$$

Inserting in (5′) and stopping at the second differences,

$$(7)\qquad l_x=l'_{z-5}+x(l'_z-l'_{z-5})+\frac{x(x-1)}{2}(l'_{z+5}-2l'_z+l'_{z-5}).$$

It should be kept in mind that in (7) $z$ is expressed in units of one year while $x$ is expressed in units of five years.

III. The area bounded by a curve, two ordinates $u_0$, $u_2$ and the axis of $x$ being divided into two equal parts by the ordinate $u_1$, an approximate value for the area is required, the distance between $u_0$ and $u_1$ being taken as the unit of length.

The general ordinate being $u_x$, the expression for the required area $\int_0^2 u_x dx$ becomes, by means of (5′),

$$\int_0^2 u_x dx=u_0\int_0^2 dx+\triangle u_0\int_0^x x dx+\frac{\triangle^2 u_0}{2}\int_0^2 x(x-1)dx=2u_0+2\triangle u_0+\tfrac{1}{3}\triangle^2 u_0.$$

Since $\triangle u_0=u_1-u_0$, $\triangle^2 u_0=u_2-2u_1+u_0$, it follows that

$$\int_0^2 u_x dx=\frac{u_0+4u_1+u_2}{3}.$$

This is known in the Calculus as Simpson's Rule, the generalization of which for the numeral solution of differential equations was made by Professor C. Runge in the *Mathematische Annalen*, volume 46.

Consider the interval $u_0 u_6$ divided into six parts by the seven equidistant ordinates $u_0$, $u_1$, $u_2$, ……………, $u_6$, the distance between the two adjacent ordinates being taken as the unit of length. The formula for the area becomes by (5′):

$$\int_0^6 u_x dx=6u_0+18\triangle u_0+27\triangle^2 u_0+24\triangle^3 u_5+\tfrac{123}{10}\triangle^4 u_0+\tfrac{33}{10}\triangle^5 u_0+\tfrac{41}{140}\triangle^6 u_0.$$

The last coefficient, $\frac{41}{140}$, differs from $\frac{42}{140}$ or $\frac{3}{10}$ by the small fraction $\frac{1}{140}$, and as from the nature of the approximation the sixth differences are supposed to be small, the error will be very slight if the last term be changed to $\frac{3}{10}\Delta^6 u_0$. By means of §2, I, it follows now that:

$$\int_0^6 u_x dx=\tfrac{3}{10}[u_0+u_2+u_4+u_6+5(u_1+u_3+u_5)+u_3].$$

This very elegant approximation for the area reads in words: Divide the area into six equal parts by means of seven ordinates and take three-tenths of the sum of the odd ordinates plus five times the sum of the even ordinates plus the middle ordinate; the distance between the ordinates being the unit of length.

The generalization of this interesting rule for the numerical solution of differential equations has not yet been made as far as I am aware.

## §5.

The term "finite integration" is employed to denote a process whose relation to $\Delta$ is analogous to the relation of integration to $D_x=\frac{d}{dx}$.

In the Calculus integration may be defined in two ways, first as an area $\int_a^x f(x)dx$ and secondly as the inverse of differentiation, *i. e.*, $D_x^{-1}f(x)$, the former being related to the latter by the equation

$$\int_a^x f(x)dx=\Big[D_x^{-1}f(x)\Big]_a^x.$$

The situation in the theory of finite differences is similar; the finite integral of $u_x$ may be also defined in two ways.

I. Let $\Delta v_x=u_x$, let $a$ be some constant and take $u_a$ as the initial term, then $v_x=S_a^x u_x$* $=u_a+u_{a+1}+\cdots\cdots\cdots+u_{x-1}$.

For $v_{x+1}=u_a+u_{a+1}+\cdots\cdots\cdots+u_{x-1}+u_x$, whence $v_{x+1}-v_x=\Delta v_x=u_x$.

II. $Su_x=[\Delta^{-1}u_x]$. The number of expressions satisfying this second definition is infinite since it includes $u_a+u_{a+1}+\cdots\cdots\cdots+u_{x+1}$ for all values of $a$. But just as in the Integral Calculus it may be shown that $S_a^x u_x=[\Delta^{-1}u_x]_a^x$.

Let $f(x)$ be a function such that $\Delta f(x)=u_x$,

$$f(a+1)-f(a)=u_a$$
$$f(a+2)-f(a+1)=u_{a+1}$$
$$\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot$$
$$f(x)-f(x-1)=u_{x-1},$$

hence, by addition, $f(x)-f(a)=S_a^x u_x$,

and from $\Delta f(x)=u_x$, which gives $f(x)=\Delta^{-1}u_x$, it is seen that $S_a^x u_x=[\Delta^{-1}u_x]_a^x$, the value for the lower limit being subtracted from the value for the upper limit. The above makes it possible to speak of definite and indefinite finite integrals.

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*We regard the use of $S$ in place of the customary $\Sigma$ as preferable, first because it corresponds closer to the $\int$ of the Integral Calculus; secondly, because it leaves the $\Sigma$ open for use in denoting ordinary summations; and thirdly, when the integration cannot be performed and the $S$ must be regarded as a functional symbol, there cannot be any confusion as may possibly occur in the case of the $\Sigma$.

III. Let $S^x u_x = f(x)$ be defined according to I and let $\Delta^{-1} u_x = f(x) + w_x$; then $u_x = \Delta f(x) + \Delta w_x = u_x + \Delta w_x$, whence $\Delta w_x = 0$; that is to say, $w =$constant, and therefore $\Delta^{-1} u_x = S^x n_x + C$.

IV. In §3 it was shown that $\Delta x^{(m)} = m x^{(m-1)}$, where $x^{(m)} = x(x-1) \ldots\ldots (x-m+1)$. This gives immediately an illustration of finite integration, namely: $S^x x^{(m)} = \frac{x^{(m+1)}}{m+1} + C$. By substituting $u_x = S v_x$ in (5′), $u_x = u_0 + x \Delta u_0 + \frac{x^{(2)} \Delta^2 u_0}{2!} + \ldots$ and writing $c = u_0$, there results

$$S v_x = c + x v_0 + \frac{x^{(2)}}{2!} \Delta v_0 + \ldots\ldots \tag{8}$$

the number of terms being finite if $v_x$ is rational and integral.

Suppose now that it be required to find the sum of $1^2 + 2^2 + \ldots\ldots + x^2$. Applying (8) and observing that $\Delta v_0 = 1$, $\Delta^2 v_0 = 2$,

$$1^2 + 2^2 + \ldots\ldots + (x-1)^2 = S x^2 = c + \frac{x(x-1)}{2} + \frac{x(x-1)(x-2)}{3!} .2.$$

When $x=2$, $c=0$, and adding $x^2$ to both sides it follows by easy reductions that $1^2 + 2^2 + \ldots\ldots + x^2 = \frac{1}{6} x(x+1)(2x+1)$.

## §6.

It sometimes happens that instead of the difference $\Delta u$ of the function $u$ being given to determine $u$, some relation between its successive differences is given. As illustration, let $\Delta^2 u_x = \frac{2}{x+1} \Delta u_x - \frac{2}{x(x+1)} \Delta_x$; it may easily be verified that the solution is $u = ax^2 + bx$.

In general, if the relation $f(x, u_x, c_1, \ldots\ldots, c_n) = 0$ is given, and if $c_1, \ldots\ldots, c_n$ are eliminated by means of the successive differences $\Delta^i f(x, u_x, c_1, \ldots\ldots, c_n) = f_i(x, u_x, c_1, \ldots\ldots, c_n) = 0$ $(i = 1 \ldots\ldots n)$, there will result a *difference equation*, $F(x, u_x, \Delta u_x, \ldots\ldots, \Delta^n u_x) = 0$, whose *complete primitive* $f(x, u_x, c_1, \ldots\ldots, c_n) = 0$ contains $n$ arbitrary constants. By §2, I, the latter equation can be replaced by one in $u_x, u_{x+1}, \ldots\ldots, u_{x+n}$; solved for $u_{x+n}$ it has the form

$$u_{x+n} = \psi(x, u_x, u_{x-1}, \ldots\ldots, u_{x+n-1}). \tag{9}$$

It will be observed that the independent variable $x$ takes on values which differ by unity, hence $u_{x+n+1} = \psi(x+1, u_{x+1}, \ldots\ldots, u_{x+n})$; eliminating $u_{x+n}$ by means of (9) the result may be written $u_{x+n+1} = \psi(x, u_x, \ldots, u_{x+n-1})$. Similarly

$$u_{x+n+m} = \psi_m(x, u_x, \ldots\ldots, u_{x+n-1}) \quad (m = 1, 2, 3, \ldots\ldots). \tag{10}$$

It is now clear that the difference equation (9) is a recursion formula for the function $u$ in terms of the $n$ arbitrary values at the points $x, x+1, \ldots\ldots, x+n-1$. Replacing $x+n+m$ by $x$ the general expression for the function is obtained. It should be noticed however that it is tacitly assumed that the function $\psi$ is finite at the points $x, x+1, \ldots\ldots, x+n+m$. Thus, if the equation is linear

$$u_{x+n} + p_x^{(1)} u_{x+n-1} + \ldots\ldots + p_x^{(n)} u_x = 0,$$

the above discussion applies only when the coefficients $p_x^{(1)}, \ldots\ldots, p_x^{(n)}$ are finite at the points $x, x+1, x+2, \ldots\ldots$ The general linear homogeneous equation of the first order may be written in either of the equivalent forms $\Delta u_x + q_x u_x = 0$, $u_{x+1} = p_x u_x$. Starting with $x = a$ and taking successive values,

$$u_{a+1} = p(a) u_a, \quad u_{a+2} = p(a+1) u_{a+1}, \ldots\ldots, u_{a+m} = p(a+m-1) u_{a+m+1},$$

this gives at once $u_{a+m} = p(a).p(a+1), \ldots\ldots, p(a+m-1) u_a$. Writing $a+m=x$ $u_a = c$, we have $u_x = c.p(x-1).p(x-2). \ldots\ldots .p(a+1).p(a)$ which is the general solution sought.

Just as in the theory of linear differential equations certain other special cases, such as equations with constant coefficients, can be integrated. For the linear equation of the $n$th order $(n>1)$ with variable coefficients no general methods are known.

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